A Recursive Sparsification of the Inverse Hodge Matrix

Alex S. Moura[1], Rodney R. Saldanha[2], Elson J. Silva[2], Adriano C, Lisboa[2],
Werley G. Facco[1], Naísses Z. Lima[2],
[1]Department of Exact Sciences – Universidade Federal dos Vales do Jequitinhonha e Mucuri
[2]Department of Electrical Engineering – Universidade Federal de Minas Gerais
alexsmoura@yahoo.com.br, rodney@cpdee.ufmg.br, elson@cpdee.ufmg.br, adriano.lisboa@enacom.com.br,
werleyfacco@yahoo.com.br, naisseszoia@gmail.com.

Abstract — **In this paper we propose a recursive technique to produce, efficiently, an approximation for the inverse of mass matrix. Using blockwise inversion a recursive algorithm allows the sparsification of the mass matrix through smaller blocks. A criterion of sparsification is proposed.**

I. INTRODUCTION

Differential forms can express Maxwell's equations in a very simplified and elegant way [1]-[2]. The use of differential forms has spread over many fields of pure and applied mathematics, and differential topology to the theory of differential equations. Differential forms are used by quantum physicists in general relativity, field theory, thermodynamics, mechanical and electromagnetism [1].

When we deal with problems of wave propagation using finite elements at time domain (FETD) we have to solve in each time step a sparse linear system defined by the insertion of the constitutive laws via the mass matrices.

The method of lumping is often used to obtain a diagonal approximation for the mass matrices in FETD [3]. Sometimes lumping destroys the matrix positive definiteness, leading to unconditional instabilities [4]. A conditionally stable, fully explicit, and sparse FETD for simplicial mesh has been achieved by approximating the mass matrix inverse of a sparse matrix [4]. This paper proposes a sparsification of the inverse matrix by recursively partitioning the original matrix into blocks.

II. MAXWELL'S EQUATIONS IN THE DISCRETE FORMS

From discrete exterior calculus we can write Maxwell's equations in their discrete form as

$$\begin{aligned} \partial_t b + \mathrm{Re} = 0 \\ -\partial_t d + R^t h = 0 \end{aligned} \tag{1}$$

where the matrix R contains only topological information on the incidence relation of the edges and faces and on their orientation, and represents a discrete curl-operator on the mesh. b , e and d , h are vectors of degrees of freedom located at faces and at edges in the primal and dual meshes, respectively. The constitutive laws are represented in discrete form as

$$\begin{aligned} d = M_1(\varepsilon)e \\ h = M_2(\nu)b \end{aligned} \tag{2}$$

where the matrices $M_i(\alpha)$ are called mass matrices .

Substituting (2) in (1) we have

$$\begin{aligned} \partial_t b + \mathrm{Re} = 0 \\ -\partial_t M_1(\varepsilon)e + R^t M_2(\nu)b = 0. \end{aligned} \tag{3}$$

The mass matrices representing the discrete form of these constitutive relations can be calculated by

$$M_p(\alpha) = \int_\Omega \alpha w_s . w_{s'} d\Omega \tag{4}$$

where α denotes some scalar field such as ε , μ , υ , etc, and w_s are p-forms of the Whitney [5]. These matrices are symmetric, positive definite and sparse. To have an explicit method, we need to find the inverse of the $M_1(\varepsilon)$ mass matrix.

III. BLOCKWISE INVERSION

Matrices can also be inverted blockwise by using the analytical inversion formula

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} Q^{-1} & -Q^{-1}BD^{-1} \\ -D^{-1}CQ^{-1} & D^{-1}\left(I + CQ^{-1}BD^{-1}\right) \end{bmatrix} \tag{5}$$

where

$$Q = \left(A - BD^{-1}C\right)$$

and A , B , C and D are sub-blocks matrices of arbitrary size (A and D are square, so they can be inverted).

In this paper we propose to apply a recursive sparsification on the mass (Hodge) matrix in order to soften and speed up the sparsification. During the process the mass matrix is recursively broken into ever smaller blocks as

$$M_1(\varepsilon) = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, D_1 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} .. D_i = \begin{bmatrix} A_{i+1} & B_{i+1} \\ C_{i+1} & D_{i+1} \end{bmatrix} \tag{6}$$

and recursively inverted by (5). An important aspect of this approach is the selection of a criterion of sparsification. Two of them are addressed in this paper.

A. *Sparsification via element thresholding*

In this criterion [4] elements are zeroed whenever

$$r_{ij} \le r \tag{7}$$

where

$$0 < r \le \min(diag)/\max(diag) \qquad (8)$$

and r_{ij} is the ratio between the absolute values of the inverse matrix D_i^{-1} and the maximum absolute value of the diagonal elements.

B. Sparsification via mean and standard deviation

A second criterion is proposed, in which elements are zeroed whenever

$$a_{lm} \le \bar{x}_i - r\sigma_i \qquad (9)$$

where $\bar{x}_i$ and σ_i are the respective mean and standard deviation of each column of D_i^{-1}, and r is a real number. This procedure is repeated until the sparsification is completed.

IV. NUMERICAL RESULTS

A example of a 3-D rectangular waveguide with perfect conducting walls and dimensions 1.0 mm $\times$ 0.5 mm $\times$ 3.0 mm is analyzed to test the efficiency and robustness of the method.

In this simulation, the parameters of the two methods are varied in order to measure the processing time to execute the sparsification methods alone and overall the leap-frog scheme.

We will now present the performance of recursive sparsification using the two criteria presented. For this we use the definition of a density matrix for a $m \times n$ matrix $[M]$. The density is defined as

$$ds([M]) = \frac{Nz}{mn}.100 \qquad (10)$$

whit Nz the number of nonzero elements.

The resulting matrices densities are also obtained, as well as the error to the analytical solution.

$$Error = \max\left(\left|\frac{E^A - E}{E^A}\right|\right) \qquad (11)$$

Solving the problem via inverse matrix Hodge we obtain a processing time of 177.49 seconds. Observing Tables I and II, the error is below 1% for a 0.0002% density, and below 0.005% for a 0.017% density. The performance is very satisfactory for both methods.

TABLE I
SPARSIFICATION VIA ELEMENT THRESHOLDING

r	Time - Seconds			Density	Error
	Sparsification	Leap-frog	total	%	%
10^{-1}	44,64	37,98	82,61	0,00020	0,83740
10^{-2}	98,47	23,65	122,12	0,00150	0,14620
10^{-3}	102,55	23,45	126,00	0,00620	0,02670
10^{-4}	119,70	12,34	132,04	0,01700	0,00430

TABLE II
SPARSIFICATION VIA MEDIA AND STANDARD DESVIATION

r	Time - Seconds			Density	Error
	Sparsification	Leap-frog	total	%	%
-0,20	53,93	15,41	69,35	0,006	0,0221
-0,10	62,10	11,15	73,25	0,0079	0,0156
-0,05	66,39	10,76	77,15	0,0099	0,0097
-0,01	73,27	11,17	84,43	0,0133	0,0067

We simulate also the method described in [4] to compare with the methods proposed in this paper. The results are shown in Table III.

TABLE III
SPARSIFICATION VIA ELEMENT THRESHOLDING WITHOUT INVERSION BY BLOCK

r	Time - Seconds			Density	Error
	Inversion and sparsification	Leap-frog	total	%	%
10^{-1}	125,50	11,50	137,00	0,00020	0,82080
10^{-2}	135,73	21,68	157,41	0,00150	0,13580
10^{-3}	139,18	27,54	166,72	0,00620	0,02310
10^{-4}	146,59	23,36	169,95	0,01710	0,00420

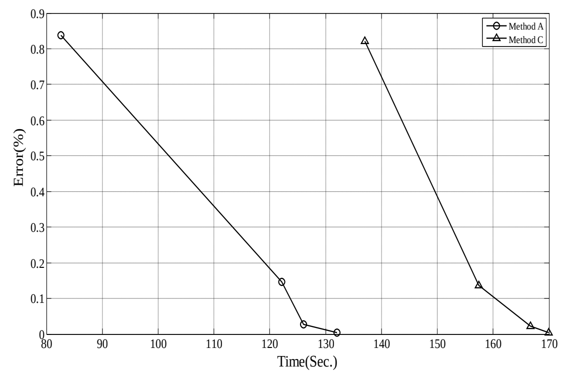

Fig. 1 Behavior (time x error) methods.
Method A – Sparsification via element thresholding
Method C – Sparsification via element thresholding without inversion by block

We conclude by the graph in Fig.1 that the proposed methods have similar performances to the method proposed by [4] in terms of accuracy, with a much shorter processing time, since we do not need to invert the Hodge matrix to apply the sparsification, we do it retreated back in the blocks of the matrices.

V. REFERENCES

[1] A. Bossavit, Computational Electromagnetism: variational formulation, complementarity, edge elements. Academic Press.,

[2] B. Donderici and F. L. Teixeira, “Mixed finite-element time-domain method for transient Maxwell equations in doubly-dispersive media,” *IEEE Microw. Theory Tech.*, vol. 56, no. 1, pp. 113–120, Jan. 2008.

[3] A. Bossavit and L. Kettunen, “Yee-like schemes on a tetrahedral mesh with diagonal lumping,” Int. J. Numer. Model., vol. 12, no. 1–2, pp. 129–142, 1999.

[4] B. He and F. L. Teixeira, “Differential forms, Galerkin duality and sparse inverse approximations in finite element solutions of Maxwell equations,” IEEE Trans. Antennas Propag., vol. 55, no. 5, pp. 1359–1368, May 2007.

[5] A. Bossavit, “Whitney forms: a class of finite elements for three-dimensional computations in electromagnetism,” Proc. Inst. Elect. Eng. A, vol. 135, no. 8, pp. 493–500, 1988.